THE ROLE OF THE KERNEL IN BONNESEN-STYLE INRADIUS INEQUALITIES

E. SAORÍN GÓMEZ

Abstract. Sharp bounds for the volume of a convex body are obtained in terms of its surface area and other quermassintegrals. These bounds are consequences of, on the one hand, inequalities for inner parallel bodies involving mixed volumes and, on the other hand, inequalities which relate a convex body to its inner parallel bodies, its kernel and its form body.

1. Introduction

Let $\mathcal{K}^n$ be the set of all convex bodies, i.e., compact convex sets in the Euclidean space $\mathbb{R}^n$, and let $\mathcal{K}^n_0$ be the subset of $\mathcal{K}^n$ consisting of all convex bodies with non-empty interior. A convex body K is called *strictly convex* if its boundary $\operatorname{bd} K$ does not contain a line segment, and *regular* if the supporting hyperplane to K at any $x \in \operatorname{bd} K$ is unique. Let B_n be the n-dimensional unit ball. The volume of a set $M \subset \mathbb{R}^n$, i.e., its n-dimensional Lebesgue measure, is denoted by $\mathrm{V}(M)$.

For two convex bodies $K \in \mathcal{K}^n$ and $E \in \mathcal{K}^n_0$ and a non-negative real number λ the *outer parallel body of K (relative to E) at distance* λ is the Minkowski sum $K + \lambda E$. For $-\mathrm{r}(K;E) \leq \lambda \leq 0$ the *inner parallel body of K (relative to E) at distance* $|\lambda|$ is the set

$$K_\lambda = \{x \in \mathbb{R}^n : |\lambda|\, E + x \subset K\},$$

where the *relative inradius* $\mathrm{r}(K;E)$ of K with respect to E is defined by

$$\mathrm{r}(K;E) = \sup\{r : \exists\, x \in \mathbb{R}^n \text{ with } x + r\, E \subset K\}.$$

When the gauge body $E = B_n$, $\mathrm{r}(K;B_n) = \mathrm{r}(K)$ is the classical inradius (see [5, p. 59]). Clearly if $\lambda = 0$ the original body K is obtained. Notice that $K_{-\mathrm{r}(K;E)}$ is the set of relative incenters of K, usually called *kernel* of K with respect to E. The dimension of $K_{-\mathrm{r}(K;E)}$ is strictly less than n (see

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[5, p. 59]). Inner parallel bodies and their properties have been studied in [3, 7, 8, 10, 11, 12, 13, 14] among others.

The so called *relative Steiner formula* [18] states that the volume of the outer parallel body $K+\lambda E$ is a polynomial of degree n in $\lambda \geq 0$,

$$\mathrm{V}(K+\lambda E)=\sum_{i=0}^{n}\binom{n}{i}\mathrm{W}_i(K;E)\lambda^i, \tag{1.1}$$

where the coefficients $\mathrm{W}_i(K;E)$ are the *relative quermassintegrals* of K, and they are particular cases of the more general *mixed volumes* (see Section 2 for definitions) for which we refer to [17, s. 5.1]. In particular, we have $\mathrm{W}_0(K;E)=\mathrm{V}(K)$ and $\mathrm{W}_n(K;E)=\mathrm{V}(E)$.

The well-known Bonnesen-Blaschke inequality for planar convex bodies $K \in \mathcal{K}^2$ and $E \in \mathcal{K}^2_0$ establishes that

$$\mathrm{W}_1(K;E)^2-\mathrm{V}(K)\mathrm{V}(E) \geq \frac{\mathrm{V}(E)^2}{4}(\mathrm{R}(K;E)-\mathrm{r}(K;E))^2 \tag{1.2}$$

where $\mathrm{R}(K;E)=1/\mathrm{r}(E;K)$ is called the circumradius of K with respect to E. Again for $E=B_n$, $\mathrm{R}(K;B_n)=\mathrm{R}(K)$ is the classical circumradius. This inequality was first proved by Bonnesen in [4] when $E=B_2$, obtaining the classical one

$$\mathrm{P}(K)^2-4\pi\mathrm{V}(K) \geq \pi^2(\mathrm{R}(K)-\mathrm{r}(K))^2,$$

where P denotes the perimeter of K. In [1] Blaschke generalized it to an arbitrary gauge body $E \in \mathcal{K}^2_0$.

Inequality (1.2) is a consequence of the more general inequality

$$\mathrm{V}(K)-2\mathrm{W}_1(K;E)x+\mathrm{V}(E)x^2 \leq 0 \tag{1.3}$$

for $\mathrm{r}(K;E) \leq x \leq \mathrm{R}(K;E)$. Equality in (1.3) holds for sausage bodies, i.e., convex bodies which are the Minkowski sum of a (possibly degenerate) segment and a dilation of E. An extension of Bonnesen's inradius inequality to higher dimensions was conjectured by Wills [19] and proved simultaneously by Bokowski [2] and Diskant [9] for $E=B_n$, and by Sangwine-Yager [15] for a general gauge body E with interior points:

$$\mathrm{V}(K)-n\mathrm{r}(K;E)\mathrm{W}_1(K;E)+(n-1)\mathrm{r}(K;E)^n\mathrm{V}(E) \leq 0. \tag{1.4}$$

In fact, in [15] Sangwine-Yager proved a much more general result, bounding the volume of every inner parallel body of K in terms of $\mathrm{V}(K)$, $\mathrm{W}_1(K;E)$, $\mathrm{W}_2(K;E)$ and some mixed volumes involving inner parallel bodies, from which (1.4) follows as a consequence. She also provided sufficient conditions for equality.

In [6], Brannen proved a strengthening of the Wills conjecture by introducing in the inequality the quermassintegrals of the kernel $K_{-\mathrm{r}(K;E)}$. This last result was improved in [13, Theorem 2.3] where also equality conditions were provided. In the proofs of these results a crucial use of the known results about inner parallel bodies is made.

In this paper we provide new inequalities for the volume of a convex body in terms of its quermassintegrals, using also the technique of inner parallel bodies. We will also show that equality conditions rely on the decomposition of the convex body through its kernel. These results will strengthen the Wills conjecture (1.4).

The paper is organized as follows. In Section 2 we provide the necessary background, such as definitions and known results which will be needed, and state the main results. Section 3 contains the proofs of these results as well as some consequences of them.

2. Background and main results

In order to state our main results we need the following definitions and notation.

For convex bodies $K_1,\dots,K_m\in\mathcal{K}^n$ and real numbers $\lambda_1,\dots,\lambda_m\geq 0$, the volume of the linear combination $\lambda_1K_1+\cdots+\lambda_mK_m$ is expressed as a polynomial of degree n in the variables $\lambda_1,\dots,\lambda_m$,

$$\mathrm{V}\bigl(\lambda_1K_1+\cdots+\lambda_mK_m\bigr)=\sum_{i_1=1}^m\cdots\sum_{i_n=1}^m\mathrm{V}(K_{i_1},\dots,K_{i_n})\lambda_{i_1}\cdots\lambda_{i_n},$$

whose coefficients $\mathrm{V}(K_{i_1},\dots,K_{i_n})$ are the *mixed volumes* of $K_1,\dots,K_m$. This formula (and hence the mixed volumes) extends the relative Steiner formula (1.1) (relative quermassintegrals).

A vector $u\in\mathbb{S}^{n-1}$ is an *r-extreme normal vector* of K, $0\leq r\leq n-1$, if it cannot be written as $u=u_1+\cdots+u_{r+2}$, with u_i linearly independent normal vectors at one and the same boundary point of K. In particular we denote the set of 0-extreme normal vectors of K by $\mathcal{U}_0(K)$. A support plane is said to be r-extreme if its outer normal vector is r-extreme. The (relative) *form body* of a convex body $K\in\mathcal{K}^n_0$ *with respect to* $E\in\mathcal{K}^n_0$, denoted by K^*, is defined as (see e.g. [8])

$$K^*=\bigcap_{u\in\mathcal{U}_0(K)}\bigl\{x:\langle x,u\rangle\leq h(E,u)\bigr\}.$$

Form bodies belong to the wider class of the so called tangential bodies. A convex body $K\in\mathcal{K}^n$ containing $E\in\mathcal{K}^n_0$ is called a *p-tangential body* of E, $p\in\{0,\dots,n-1\}$, if each $(n-p-1)$-extreme support plane of K supports E. Notice that the construction of the form body of a convex body K with respect to $E\in\mathcal{K}^n_0$ yields that it is always an $(n-1)$-tangential body of E. There is a very close connection between inner parallel bodies and tangential bodies for which we refer to [17, pp. 136–137].

We will need some relations between inner parallel bodies, the form body and the kernel of a convex body. From now on and for the sake of brevity we denote $\mathrm{r}(K;E)$ by r.

From the definition of inner parallel body for every $-\mathrm{r}\leq\lambda\leq 0$, it follows that $K_{-\mathrm{r}}+(\mathrm{r}+\lambda)E\subseteq K_\lambda$ with equality for all λ if and only if $K=K_{-\mathrm{r}}+\mathrm{r}E$.

In [14, Lemma 4.4] it is shown that $\operatorname{cl}\mathcal{U}_0(K_\lambda) \subseteq \mathcal{U}_0(K)$. Thus, for $-\mathrm{r} < \lambda \leq 0$ the relation $K_\lambda^* \supseteq K^*$ holds, where K_λ^* is the form body of K_λ. Moreover, it is known (see [14, Lemma 4.8]) that it always holds

$$K \supseteq K_\lambda + |\lambda|\, K^*$$

for any $K \in \mathcal{K}^n$, $E \in \mathcal{K}_0^n$ and all $-\mathrm{r} \leq \lambda \leq 0$.

In [13, Theorem 2.2] convex bodies $K \in \mathcal{K}^n$ satisfying $K = K_\lambda + |\lambda|\, K^*$ for every $-\mathrm{r} \leq \lambda \leq 0$ are characterized as special tangential bodies of $K_{-\mathrm{r}} + \mathrm{r}E$. We include the result here for completeness.

(2.1) Let $E \in \mathcal{K}_0^n$ be regular. Then $K = K_\lambda + |\lambda|\, K^*$ for every $-\mathrm{r} \leq \lambda \leq 0$ if and only if K is a tangential body of $K_{-\mathrm{r}} + \mathrm{r}E$ so that for all $-\mathrm{r} \leq \lambda \leq 0$, the condition $\mathcal{U}_0(K) = \mathcal{U}_0(K_\lambda + K^*)$ is satisfied.

This result also shows that in order to have the decomposition $K = K_\lambda + |\lambda|\, K^*$, for $-\mathrm{r} \leq \lambda \leq 0$, it is enough to have it just for $\lambda = -\mathrm{r}$.

We have the following inclusions which we will need for the proofs of the main results

$$K_{-\mathrm{r}} + (\mathrm{r} + \lambda)K^* \subseteq K_{-\mathrm{r}} + (\mathrm{r} + \lambda)K_\lambda^* \subseteq K_\lambda \tag{2.2}$$

for $-\mathrm{r} < \lambda \leq 0$.

It is known that the function $\mathrm{V}(\lambda) := \mathrm{V}(K_\lambda)$ defined on $-\mathrm{r} \leq \lambda \leq 0$ is differentiable and $\mathrm{V}'(\lambda) = n\mathrm{W}_1(K_\lambda; E)$. Its corresponding integral form

$$\mathrm{V}(K) = n \int_{-\mathrm{r}}^{0} \mathrm{W}_1(K_\lambda; E)\, d\lambda \tag{2.3}$$

is a key element in the proofs of the results.

First, we will prove the following upper bound for the volume of a convex body in terms of the first quermassintegral and a finite sum of mixed volumes involving K, its kernel $K_{-\mathrm{r}}$, its form body K^* and the gauge body E. This result strengthens the Wills conjecture (1.4) (see Remark 3.1).

Theorem 2.1. *Let $K \in \mathcal{K}^n$, $E \in \mathcal{K}_0^n$ and r be the relative inradius of K with respect to E. Then*

$$\begin{aligned} \mathrm{V}(K) &\leq n\mathrm{W}_1(K; E)\mathrm{r} \\ &\quad - n \sum_{k=0}^{n-2} \sum_{j=0}^{k} \frac{\binom{k}{j}}{c_{k,j}} \mathrm{V}\big(K_{-\mathrm{r}}[j], K^*[k-j+1], K[n-k-2], E\big)\mathrm{r}^{k-j+2}, \end{aligned} \tag{2.4}$$

where $c_{k,j} = (k-j+1)(k-j+2)$. If $K = K_{-\mathrm{r}} + \mathrm{r}K^*$ equality holds. If E is regular and strictly convex and equality holds then K is a tangential body of $K_{-\mathrm{r}} + \mathrm{r}E$.

This result is a direct consequence of the following more general one.

Theorem 2.2. *Let $K\in\mathcal{K}^n$, $E\in\mathcal{K}^n_0$ and r be the relative inradius of K with respect to E. Then for $-\mathrm{r}\leq\lambda\leq 0$*

$$\begin{aligned}\mathrm{V}(K_\lambda)\leq{}& n\mathrm{W}_1(K;E)(\mathrm{r}+\lambda)\\ &+n\sum_{k=0}^{n-2}\sum_{j=0}^{k}\left[\frac{\binom{k}{j}}{c_{k,j}}\mathrm{V}\bigl(K_{-\mathrm{r}}[j],K^*[k-j+1],K[n-k-2],E\bigr)\right.\\ &\left.\bigl[(k-j+1)\lambda-r\bigr](\mathrm{r}+\lambda)^{k-j+1}\right].\end{aligned}\tag{2.5}$$

If $K=K_{-\mathrm{r}}+\mathrm{r}K^*$, then equality holds. If E is regular and strictly convex and equality holds for some $-\mathrm{r}<\lambda\leq 0$, then K is a tangential body of $K_{-\mathrm{r}}+\mathrm{r}E$.

Notice that Theorem 2.1 is obtained by taking $\lambda=0$ in Theorem 2.2.

Next we use a technique used by Diskant [9] and Brannen [6] to prove the following inequality.

Theorem 2.3. *Let $K\in\mathcal{K}^n$, $E\in\mathcal{K}^n_0$ and r be the relative inradius of K with respect to E. Then*

$$\mathrm{V}(K)\leq n\mathrm{W}_1(K;E)\mathrm{r}-n\sum_{j=0}^{n-1}\binom{n-1}{j}\frac{j}{j+1}\mathrm{V}\bigl(K_{-\mathrm{r}}[n-j-1],K^*[j],E\bigr)\mathrm{r}^{j+1}.\tag{2.6}$$

If $K=K_{-\mathrm{r}}+\mathrm{r}K^*$ equality holds.

This result is a consequence, by taking $\lambda=0$, of the following more general result, which provides bounds for the volume of the whole family of inner parallel bodies of K.

Theorem 2.4. *Let $K\in\mathcal{K}^n$, $E\in\mathcal{K}^n_0$ and r be the relative inradius of K with respect to E. Then for all $-\mathrm{r}<\lambda<0$*

$$\begin{aligned}\mathrm{V}(K_\lambda)\leq{}& n\mathrm{W}_1(K;E)(\mathrm{r}+\lambda)\\ &+n\sum_{j=0}^{n-1}\binom{n-1}{j}\mathrm{V}\bigl(K_{-\mathrm{r}}[n-j-1],K^*[j],E\bigr)\left[\frac{(\mathrm{r}+\lambda)^{j+1}}{j+1}-\mathrm{r}^j(r+\lambda)\right].\end{aligned}\tag{2.7}$$

If $K=K_{-\mathrm{r}}+\mathrm{r}K^*$ we get equality.

3. Proofs of the main results

For the proof of Theorem 2.2, we will need the following inequality contained in [13, Theorem 2.3]. For $i=0,\dots,n-1$ and $-\mathrm{r}\leq\lambda\leq 0$,

$$\mathrm{W}_i(K_\lambda;E)\leq\mathrm{W}_i(K;E)-|\lambda|\sum_{k=0}^{n-i-1}\mathrm{V}\bigl(K_\lambda[k],K[n-i-k-1],K^*,E[i]\bigr).\tag{3.1}$$

If $K=K_{-\mathrm{r}}+\mathrm{r}K^*$ then equality holds in all the inequalities. Conversely, if E is regular and strictly convex and equality holds in (3.1) for some $i\in\{0,\dots,n-1\}$ then K is a tangential body of $K_{-\mathrm{r}}+\mathrm{r}E$.

Now we deal with the proofs of the main results.

Proof of Theorem 2.2. First we consider inequality (3.1) for the case of W_1, i.e., for every $-\mathrm{r}\le\mu\le 0$

$$\mathrm{W}_1(K_\mu;E)\le \mathrm{W}_1(K;E)-|\mu|\sum_{k=0}^{n-2}\mathrm{V}\big(K_\mu[k],K[n-k-2],K^*,E\big).$$

Using the integral form of the volume (2.3), we can integrate the inequality with respect to μ and obtain that

(3.2)
$$\begin{aligned}\frac{1}{n}\mathrm{V}(K_\lambda)&=\int_{-\mathrm{r}}^{\lambda}\mathrm{W}_1(K_\mu;E)d\mu\\ &\le\int_{-\mathrm{r}}^{\lambda}\left(\mathrm{W}_1(K;E)-|\mu|\sum_{k=0}^{n-2}\mathrm{V}\big(K_\mu[k],K[n-k-2],K^*,E\big)\right)d\mu\\ &\le \mathrm{W}_1(K;E)(\mathrm{r}+\lambda)+\int_{-\mathrm{r}}^{\lambda}\left(\mu\sum_{k=0}^{n-2}\mathrm{V}\big(K_\mu[k],K[n-k-2],K^*,E\big)\right)d\mu.\end{aligned}$$

Notice that inside the integral we have mixed volumes depending on μ where at least three different convex bodies are involved. In order to bound these ones we observe that (2.2) and the monotonicity of mixed volumes (see e.g. [17, p. 277]) yield

$$\mathrm{V}\big(K_\mu[k],K[n-k-2],K^*,E\big)\ge\mathrm{V}\big(K_{-\mathrm{r}}+(\mathrm{r}+\mu)K^*[k],K[n-k-2],K^*,E\big)$$

and so, using the linearity of the mixed volumes (see e.g. [17, p. 279]), the integral above can be bounded as follows:

$$\begin{aligned}&\int_{-\mathrm{r}}^{\lambda}\left(\mu\sum_{k=0}^{n-2}\mathrm{V}\big(K_\mu[k],K[n-k-2],K^*,E\big)\right)d\mu\\ &\quad\le\int_{-\mathrm{r}}^{\lambda}\left(\mu\sum_{k=0}^{n-2}\mathrm{V}\big(K_{-\mathrm{r}}+(\mathrm{r}+\mu)K^*[k],K[n-k-2],K^*,E\big)\right)d\mu\\ &\quad=\int_{-\mathrm{r}}^{\lambda}\sum_{k=0}^{n-2}\sum_{j=0}^{k}\binom{k}{j}\mu(\mathrm{r}+\mu)^{k-j}\mathrm{V}\big(K_{-\mathrm{r}}[j],K^*[k-j],K[n-k-2],K^*,E\big)d\mu\\ &\quad=\sum_{k=0}^{n-2}\sum_{j=0}^{k}\binom{k}{j}\mathrm{V}\big(K_{-\mathrm{r}}[j],K^*[k-j+1],K[n-k-2],E\big)\int_{-\mathrm{r}}^{\lambda}\mu(\mathrm{r}+\mu)^{k-j}d\mu.\end{aligned}$$

Since

$$\int_{-\mathrm{r}}^{\lambda}\mu(\mathrm{r}+\mu)^{k-j}d\mu=\frac{\big[(k-j+1)\lambda-r\big](\mathrm{r}+\lambda)^{k-j+1}}{c_{k,j}},$$

plugging this in (3.2) we get the announced bound for the volume.

If $K = K_{-\mathrm{r}} + \mathrm{r}K^*$, then it is clear that equality holds, since in this case $K_\lambda = K_{-\mathrm{r}} + (\mathrm{r}+\lambda)K^*$. If E is regular and strictly convex and equality holds in (2.5) for some $-\mathrm{r} < \lambda \le 0$, then it also holds in (3.1). Thus, from [13, Theorem 2.3] it follows that K is a tangential body of $K_{-\mathrm{r}} + \mathrm{r}E$. $\square$

As a corollary of inequality (2.4) we obtain the following strengthening of Wills' conjecture (1.4).

Corollary 3.1. *Let $E \in \mathcal{K}_0^n$, $K \in \mathcal{K}^n$ and r be the relative inradius of K with respect to E. Then*

$$\mathrm{V}(K) \le n\mathrm{W}_1(K;E)\mathrm{r} - n\sum_{k=0}^{n-2}\mathrm{W}_{k+2}(K;E)\frac{\mathrm{r}^{k+2}}{(k+1)(k+2)}. \tag{3.3}$$

Proof. Using Theorem 2.1, since $E \subset K^*$ we obtain that

$$\begin{aligned}\mathrm{V}(K) \le{}& n\mathrm{W}_1(K;E)\,\mathrm{r}\\ &- n\sum_{k=0}^{n-2}\sum_{j=0}^{k}\binom{k}{j}\mathrm{V}\big(K_{-\mathrm{r}}[j], E[k-j+2], K[n-k-2]\big)\frac{\mathrm{r}^{k-j+2}}{c_{k,j}}.\end{aligned}$$

Taking just the summands corresponding to $j = 0$ for every k we get the desired bound for the volume:

$$\begin{aligned}\mathrm{V}(K) &\le n\mathrm{W}_1(K;E)\mathrm{r} - n\sum_{k=0}^{n-2}\mathrm{V}\big(K[n-k-2], E[k+2]\big)\frac{\mathrm{r}^{k+2}}{(k+1)(k+2)}\\ &= n\mathrm{W}_1(K;E)\mathrm{r} - n\sum_{k=0}^{n-2}\mathrm{W}_{k+2}(K;E)\frac{\mathrm{r}^{k+2}}{(k+1)(k+2)}.\end{aligned}$$ $\square$

Remark 3.1. *Observe that since $\mathrm{r}E \subset K$, the monotonicity of the mixed volumes yields that $\mathrm{r}^n\mathrm{V}(E) \le \mathrm{r}^i\mathrm{W}_i(K;E) \le \mathrm{V}(K)$ and thus the above sum can be bounded as*

$$\mathrm{r}^n\mathrm{V}(E)\frac{n-1}{n} \le \sum_{k=0}^{n-2}\mathrm{W}_{k+2}(K;E)\frac{\mathrm{r}^{k+2}}{(k+1)(k+2)} \le \mathrm{V}(K)\frac{n-1}{n}.$$

Then it is clear that inequality (3.3) strengthens Wills' conjecture inequality:

$$\begin{aligned}0 &\ge \mathrm{V}(K) - n\mathrm{W}_1(K;E)\mathrm{r} + n\sum_{k=0}^{n-2}\mathrm{W}_{k+2}(K;E)\frac{\mathrm{r}^{k+2}}{(k+1)(k+2)}\\ &\ge \mathrm{V}(K) - n\mathrm{W}_1(K;E)\mathrm{r} + n\mathrm{r}^n\mathrm{V}(E)\sum_{k=0}^{n-2}\frac{1}{(k+1)(k+2)}\\ &= \mathrm{V}(K) - n\mathrm{W}_1(K;E)\mathrm{r} + (n-1)\mathrm{r}^n\mathrm{V}(E).\end{aligned}$$

Notice that if $K = \mathrm{r}K^*$ (in particular, in this case $K_{-\mathrm{r}}$ is a point) then we have equality. The condition $K = \mathrm{r}K^*$ is satisfied if and only if K is a tangential body of E.

Next we will prove Theorem 2.4. Its proof follows the ideas of the proof of [6, Theorem 4] which will be obtained as a corollary.

Proof of Theorem 2.4. First we prove the inequality

$$\mathrm{W}_1(K;E)-\mathrm{W}_1(K_{-\mathrm{r}}+\mathrm{r}K^*;E)\geq \mathrm{W}_1(K_\lambda;E)-\mathrm{W}_1(K_{-\mathrm{r}}+(\mathrm{r}+\lambda)K^*;E). \tag{3.4}$$

Writing $K_{-\mathrm{r}}+\mathrm{r}K^*=K_{-\mathrm{r}}+(\mathrm{r}+\lambda)K^*+|\lambda|\,K^*$, we can compute the quermassintegral $\mathrm{W}_1(K_{-\mathrm{r}}+\mathrm{r}K^*;E)$ as follows:

$$\begin{aligned}&\mathrm{W}_1(K_{-\mathrm{r}}+\mathrm{r}K^*;E)\\&\qquad=\sum_{j=0}^{n-1}\binom{n-1}{j}\mathrm{V}\big(K_{-\mathrm{r}}+(\mathrm{r}+\lambda)K^*[j],K^*[n-j-1],E\big)\,|\lambda|^{n-j-1}.\end{aligned}$$

The right hand side can be rewritten as

$$\begin{aligned}&\mathrm{W}_1(K_{-\mathrm{r}}+(\mathrm{r}+\lambda)K^*;E)\\&\qquad+\sum_{j=0}^{n-2}\binom{n-1}{j}\mathrm{V}\big(K_{-\mathrm{r}}+(\mathrm{r}+\lambda)K^*[j],K^*[n-j-1],E\big)\,|\lambda|^{n-j-1}.\end{aligned}$$

Thus, the following holds

$$\begin{aligned}&\mathrm{W}_1(K;E)+\mathrm{W}_1(K_{-\mathrm{r}}+(\mathrm{r}+\lambda)K^*;E)-\mathrm{W}_1(K_{-\mathrm{r}}+\mathrm{r}K^*;E)\\&\quad=\mathrm{W}_1(K;E)-\sum_{j=0}^{n-2}\binom{n-1}{j}\mathrm{V}\big(K_{-\mathrm{r}}+(\mathrm{r}+\lambda)K^*[j],K^*[n-j-1],E\big)\,|\lambda|^{n-j-1},\end{aligned}$$

and hence it is enough to prove that

$$\begin{aligned}&\mathrm{W}_1(K;E)-\sum_{j=0}^{n-2}\binom{n-1}{j}\mathrm{V}\big(K_{-\mathrm{r}}+(\mathrm{r}+\lambda)K^*[j],K^*[n-j-1],E\big)\,|\lambda|^{n-j-1}\\&\qquad\geq \mathrm{W}_1(K_\lambda;E).\end{aligned} \tag{3.5}$$

The monotonicity of the mixed volumes together with (2.2) yield

$$\begin{aligned}&\mathrm{W}_1(K;E)-\sum_{j=0}^{n-2}\binom{n-1}{j}\mathrm{V}\big(K_{-\mathrm{r}}+(\mathrm{r}+\lambda)K^*[j],K^*[n-j-1],E\big)\,|\lambda|^{n-j-1}\\&\quad\geq \mathrm{W}_1(K;E)-\sum_{j=0}^{n-2}\binom{n-1}{j}\mathrm{V}\big(K_\lambda[j],K^*[n-j-1],E\big)\,|\lambda|^{n-j-1}\\&\quad=\mathrm{W}_1(K;E)+\mathrm{W}_1(K_\lambda;E)\\&\qquad-\sum_{j=0}^{n-1}\binom{n-1}{j}\mathrm{V}\big(K_\lambda[j],K^*[n-j-1],E\big)\,|\lambda|^{n-j-1}\\&\quad=\mathrm{W}_1(K;E)+\mathrm{W}_1(K_\lambda;E)-\mathrm{W}_1(K_\lambda+|\lambda|\,K^*;E)\geq \mathrm{W}_1(K_\lambda;E)\end{aligned}$$

because $K \supseteq K_\lambda + |\lambda| K^*$. This proves (3.5) and hence (3.4). Notice that if $K = K_{-\mathrm{r}} + \mathrm{r}K^*$ (cf. [13, Theorem 2.2]), then inequality (3.4) becomes an equality.

Now, integrating (3.4) and using (2.3) we get, for $-\mathrm{r} \le \lambda \le 0$, that

$$\begin{aligned}\frac{1}{n}\mathrm{V}(K_\lambda) &= \int_{-\mathrm{r}}^{\lambda} \mathrm{W}_1(K_\mu; E)\,d\mu \\ &\le \int_{-\mathrm{r}}^{\lambda} \Big[\mathrm{W}_1(K;E) + \mathrm{W}_1(K_{-\mathrm{r}} + (\mathrm{r}+\mu)K^*;E) - \mathrm{W}_1(K_{-\mathrm{r}} + \mathrm{r}K^*;E)\Big]d\mu.\end{aligned}$$

Using again the linearity of mixed volumes for $\mathrm{W}_1(K_{-\mathrm{r}} + \mathrm{r}K^*; E)$ and $\mathrm{W}_1\big(K_{-\mathrm{r}} + (\mathrm{r}+\lambda)K^*; E\big)$, the previous inequality becomes

$$\begin{aligned}\frac{1}{n}\mathrm{V}(K_\lambda) \le{}& (\mathrm{r}+\lambda)\mathrm{W}_1(K;E) \\ &- (\mathrm{r}+\lambda)\sum_{j=0}^{n-1}\binom{n-1}{j}\mathrm{V}\big(K_{-\mathrm{r}}[j], K^*[n-j-1], E\big)\mathrm{r}^{n-j-1} \\ &+ \sum_{j=0}^{n-1}\binom{n-1}{j}\mathrm{V}\big(K_{-\mathrm{r}}[j], K^*[n-j-1], E\big)\int_{-\mathrm{r}}^{\lambda}(\mathrm{r}+\mu)^{n-j-1}d\mu.\end{aligned}$$

Computing the integral it follows that

$$\begin{aligned}\frac{1}{n}\mathrm{V}(K_\lambda) \le{}& \mathrm{W}_1(K;E)(\mathrm{r}+\lambda) \\ &+ \sum_{j=0}^{n-1}\binom{n-1}{j}\mathrm{V}\big(K_{-\mathrm{r}}[j], K^*[n-j-1], E\big)\left[\frac{(\mathrm{r}+\lambda)^{n-j}}{n-j} - (\mathrm{r}+\lambda)\mathrm{r}^{n-j-1}\right],\end{aligned}$$

which ends the proof of (2.7).

Sufficient conditions for the equality case in (3.4) provide sufficient conditions for equality in (2.7). Thus, if $K = K_{-\mathrm{r}} + \mathrm{r}K^*$, we get equality for all $-\mathrm{r} \le \lambda \le 0$. □

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Institut für Algebra und Geometrie, Otto-von-Guericke Universität Magdeburg, Universitätsplatz 2, DE-39106 Magdeburg, Germany
E-mail address: eugenia.saorin@ovgu.de